Theory of ion–hydrogen collisions in a uniform magnetic field

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Abstract. The present work provides a theoretical treatment of transitions between the Zeeman magnetic sublevels \( m \) of atomic hydrogen induced by ion collisions in the presence of a uniform magnetic field. This process is important in discussing the polarization properties of the optical emission in plasmas accompanied by an external magnetic field. Owing to the long-range dipole interaction, \( n^2 \) sublevels with the same principal quantum number \( n \) are strongly coupled. However, by using a peculiar symmetry property of atomic hydrogen, the number of coupled channels is reduced from \( n^2 \) to \( n \). Furthermore, a useful scaling law is found.

1. Introduction

Recently, it has become very important to understand the polarization properties of the optical emission of astrophysical and laboratory plasmas (Kazantsev 1983, Kazantsev et al 1988, Fujimoto et al 1992). Measurements of the degree of polarization provide us with important information on these plasmas. The presence of anisotropic properties or non-thermal particles in the plasmas (or the absorption of polarized photons) leads to the non-uniform population of the magnetic sublevels of degenerate atomic states (alignment). It is the origin of the polarized optical emission of the plasmas. On the other hand, the non-uniform population is smoothed out by collisions with particles (disalignment). Thus, the observed degree of polarization is determined by the balance of the processes that produce alignment or disalignment.

Collisional transitions between the degenerate magnetic sublevels are an important disalignment mechanism. As in tokamaks or solar flares, an external magnetic field sometimes acts on the plasmas. A magnetic field causes the Zeeman splitting of the degenerate magnetic sublevels. Since the collision cross section depends on the energy difference between the magnetic sublevels, the magnetic field will affect the disalignment process (e.g. Gay and Schneider 1979). Despite its importance, this type of problem has not yet been studied sufficiently.

In the present study, we consider the collisions between an excited hydrogen atom and an ion in a uniform magnetic field. We focus on the transition between the Zeeman magnetic sublevels \( m \) with the same principal quantum number \( n \). Because of the long-range ion–dipole interaction, all the \( n^2 \) sublevels \( (l, m) \) with the same \( n \) are strongly coupled during the collision \( (l \) being the orbital angular momentum quantum number). It seems to be difficult to do the numerical calculation for high \( n \) states. However, by introducing the method of Demkov et al (1970, 1974), we find that the number of coupled channels can be reduced from \( n^2 \) to \( n \). Furthermore,
a useful scaling formula is satisfied; i.e. the calculations are made once for reduced
quantities and the results can then be scaled to an arbitrary field strength.

The present subject is directly connected with the theoretical study of state selec-
tive charge transfer (Thorson and Choi 1984, Kato et al 1991) that is important in
plasmas accompanied by magnetic fields. Charge transfer from a hydrogen atom to
a highly charged ion often occurs to a high $n$ hydrogenic level. Therefore, we must
solve the mixing within the Zeeman manifold due to the long-range dipole coupling
after the charge transfer. This is just the subject that we shall discuss.

2. Theory

We consider weak magnetic fields. Therefore, we can assume that the energy splitting
is determined by the linear Zeeman effect and that the magnetic field does not affect
the ion motion during the collision.

In the present study, we neglect the motional Stark effect which takes place when
the hydrogen atom moves across the magnetic field. As shown in the appendix, this
effect is negligible if $3nV \ll 1$ where $V$ is the hydrogen velocity (in au) perpendicular
to the magnetic field. (However, the inclusion of the motional Stark effect is possible
in the following formulation. See the appendix.)

Owing to the small energy difference between the magnetic sublevels, the ionic-
dipole interaction plays a dominant role and the transition occurs in distant collisions.
Hence, we neglect other short-range interactions and keep only the asymptotic form
of the dipole interaction. We denote the relative coordinate between the hydrogen
atom and the ion with charge $+Z$ by $R(t)$, which is specified by a classical trajectory
as a function of time $t$. Then, the time-dependent Schrödinger equation is given by

$$i\frac{\partial}{\partial t} |\Psi\rangle = H |\Psi\rangle$$  \hspace{1cm} (1)

with

$$H = -\frac{Z\hat{R} \cdot \hat{r}}{R^2} + B \cdot \hat{l} + H_0$$

where $\hat{r}$ and $\hat{l}$ are, respectively, the coordinate and orbital angular momentum op-
erators of the electron measured from the proton; $H_0$ is the Hamiltonian of atomic
hydrogen; and $B = (0, 0, B)$ is the magnetic field. The $z$ axis is taken along the
field direction. Atomic units have been used. The conversion rate of $B$ is

$$B = 2.127 \times 10^{-6} B \text{ (in Tesla)}.$$  

We have neglected the very weak spin–orbit interaction. This is acceptable when the
Zeeman splitting is larger than the energy difference of the fine structure.

As a solution of the Schrödinger equation, the wavefunction is usually expanded
in terms of the eigenfunction of $l^2$ and $l_z$. In the present study, however, we
introduce a different way, which is better because the number of channels in the
resulting coupled equations can be significantly reduced. For this purpose, we use the
symmetry property of atomic hydrogen.
2.1. Peculiar symmetry of atomic hydrogen

Atomic hydrogen possesses its own peculiar symmetry because the angular momentum and also the Lunge–Renz vector commute with the Hamiltonian $H_0$. The Lunge–Renz vector is defined by

$$\alpha = (-2H_0)^{-1/2}[\frac{1}{2}(\mathbf{p} \times \mathbf{l} - \mathbf{l} \times \mathbf{p}) - r]$$

where $\mathbf{p}$ is the electron momentum. The operators $\mathbf{l}$ and $\alpha$ generate the four-dimensional rotational group $O(4)$ (Biedenharn 1961). To construct the representation of $O(4)$, it is convenient to use the fact that $O(4)$ is isomorphic to $O(3) \times O(3)$. The vector operators $j_1$ and $j_2$, defined by

$$j_1 = \frac{1}{2}(l + \alpha) \quad \text{and} \quad j_2 = \frac{1}{2}(l - \alpha)$$

(2)

commute with each other and obey the commutation rules of angular momenta. The eigenvector of atomic hydrogen is given by

$$|j_1 k_1 j_2 k_2\rangle = |j_1 k_1\rangle |j_2 k_2\rangle$$

(3)

where ($i = 1, 2$)

$$j_i^2 |j_i k_i\rangle = j_i(j_i + 1) |j_i k_i\rangle$$

(4)

$$\langle j_i | j_i k_i\rangle = k_i |j_i k_i\rangle$$

(5)

with

$$j_1 = j_2 = j \equiv \frac{1}{2}(n - 1) \quad \text{and} \quad k_1 + k_2 = m$$

(6)

$m_n$ being the quantum number of $l_z$. The quantum number $k_i$ takes the values of $-j, -j + 1, \ldots, j - 1, j$. Here, the $z$ axis has been chosen as the quantization axis. However, it should be noted that an arbitrary axis can be taken separately each for the quantization of $j_1$ and $j_2$. We will use this fact later.

Since we treat the transition between the magnetic sublevels within the same $n$, we consider equation (1) in a subspace of given $n$. In this case, the operator $r$ can be replaced by

$$r = -\frac{3}{2} n \alpha.$$  

(7)

This relation is directly shown by comparing the matrix elements of the coordinate vector $r$ (Bethe and Salpeter 1977) and those of the Lunge–Renz vector $\alpha$ (Biedenharn 1961). Demkov et al (1970, 1974) introduced the identity (7) to calculate the energy levels of a hydrogen atom in crossed electric and magnetic fields and to study the collisions of a hydrogenic ion by a charged particle in the absence of an external field. This method has wider applicability. It is also very useful for the present problem.
2.2. Scaling

Before considering a decoupling method, we discuss the scaling law of the present system. Introducing the following reduced quantities:

$$ \tau = Bt \quad \text{and} \quad X = \left( \frac{2B}{3nZ} \right)^{1/2} R $$

and substituting the identity (7) into equation (1), we have

$$ i \frac{\partial}{\partial \tau} |\Psi\rangle = h |\Psi\rangle $$

with

$$ h = \frac{\hat{R} \cdot a}{X^2} + \hat{B} \cdot l. $$

(10)

Here, we have omitted the $H_0$ term, which is merely a constant ($= -\frac{1}{2}n^{-2}$).

In solving the Schrödinger equation (9), we must give a classical trajectory for the relative motion. In the present case, we adopt a linear trajectory with a constant velocity $v$. As seen in the calculation, the major part of the contribution to the cross section comes from the collision with an impact parameter much larger than 100 au when $B = 1$ T, $n = 3$, and $Z = 1$ (see figure 3). Therefore, the assumption of a linear trajectory is fairly valid in the calculation of the cross section.

Let $\beta$ denote the reduced velocity defined by

$$ \beta = \left( \frac{2}{3nZB} \right)^{1/2} v. $$

(11)

Then, we can easily verify that the transition probability calculated by equation (9) with a linear trajectory is dependent only on $\beta$ but not on $B$ or $Z$ (cf Takayanagi 1978). Therefore, we can solve equation (9) regardless of the field strength; and the results are scaled to an arbitrary field strength by using (8) and (11).

2.3. Separability of the Schrödinger equation

From (2), equation (10) becomes

$$ h = w_1 \cdot j_1 + w_2 \cdot j_2 $$

(12)

where

$$ w_1 = \hat{B} + \frac{1}{X^2} \hat{R} \quad \text{and} \quad w_2 = \hat{B} - \frac{1}{X^2} \hat{R}. $$

(13)

It is evident that the Hamiltonian $h$ is separable. Putting $|\Psi\rangle = |\Psi_1\rangle |\Psi_2\rangle$ in equation (9) where $j_i$ ($i = 1, 2$) operates only on $|\Psi_i\rangle$, we have ($i = 1, 2$)

$$ i \frac{\partial}{\partial \tau} |\Psi_i\rangle = h_i |\Psi_i\rangle $$

(14)

with

$$ h_i = w_i \cdot j_i. $$

(15)
2.4. Coupled equations

We diagonalize the Hamiltonian $h_i$. This can be easily done by taking the quantization axes of $j_1$ and $j_2$, respectively, along $\hat{w}_1$ and $\hat{w}_2$, i.e.

$$h_i |\lambda_i\rangle = w_i \lambda_i |\lambda_i\rangle$$  \hspace{1cm} (16)

where $\lambda_i$ is the quantum number of the $\hat{w}_i$ component of $j_i$. In the limit of $X \to \infty$ (i.e. before and after the collision), both $w_1$ and $w_2$ are identical to $\hat{B}$. Hence,

$$|\lambda_i\rangle \to |j_i k_i\rangle \quad \text{and} \quad \lambda_i \to k_i \quad \text{as} \quad X \to \infty.$$  \hspace{1cm} (17)

In this limit, the sum of $\lambda_1$ and $\lambda_2$ is regarded as the quantum number of $l_z$. However, this is not true for a finite value of $X$ since the quantization axes are different.

We consider the special case that the vector $\hat{R}$ lies in the $xz$ plane. Let $\alpha_i$ ($i = 1, 2$) denote the polar angle of $w_i$. Since $|\lambda_i\rangle$ is also an eigenfunction of $j_i$, we have, by a rotation of the quantization axis,

$$|\lambda_i\rangle = \sum_k d_{k_i \lambda_i \lambda_i}(\alpha_i)|j_i k_i\rangle$$  \hspace{1cm} (18)

where $d_{k_i \lambda_i \lambda_i}(\alpha)$ is the matrix element of a finite rotation (Rose 1957).

We expand the wavefunction $\Psi_i$ in terms of $|\lambda_i\rangle$ as

$$|\Psi_i(\tau)\rangle = \sum_{\lambda_i} C_{\lambda_1, \lambda_i}(\tau)|\lambda_i\rangle \exp \left( -i \lambda_i \int^{\tau} w_i \, d\tau' \right).$$  \hspace{1cm} (19)

Inserting this equation into (14) provides

$$\frac{dC_{\lambda_1, \lambda_i}}{d\tau} = - \sum_{\lambda_i' \neq \lambda_i} \langle \lambda_i | \frac{d}{d\tau} | \lambda_i' \rangle C_{\lambda_i', \lambda_i} \exp \left( i(\lambda_i - \lambda_i') \int^{\tau} w_i \, d\tau' \right).$$  \hspace{1cm} (20)

If $\hat{R}$ lies in the $xz$ plane, the explicit form of the non-adiabatic coupling matrix element is easily obtained from (18) as

$$\langle \lambda_i | \frac{d}{d\tau} | \lambda_i' \rangle = \mp \frac{1}{2} \frac{d\alpha_i}{d\tau} \sqrt{(j \mp \lambda_i')(j \pm \lambda_i' + 1)} \delta_{\lambda_i, \lambda_i' \pm 1}.$$  \hspace{1cm} (21)

We solve the coupled equations (20) numerically. The number of coupled channels in (20) is $n$, which is significantly reduced from the total number of channels ($n^2$).

2.5. Cross sections

We are considering a weak magnetic field so that it does not affect the ion motion during the collision. Therefore, we can calculate the cross section in the usual manner. However, it should be noted that the cross section also has a dependence on the incident direction of the ion relative to the magnetic field. This can be easily understood by looking at equation (13), which shows that the adiabatic state $|\lambda_i\rangle$ is
clearly dependent on the orientation of $R$ relative to $B$. Consequently, the cross section is given as a function of the incident velocity vector $\beta$.

We introduce the impact parameter vector $b$ (in the reduced units) which is perpendicular to the incident direction of the ion. In the present case, the transition for each collision depends not only on the magnitude of $b$ but also on its orientation. Hence, at each given $b$, imposing the initial condition by

$$C^{(\lambda_0^i)}_{\lambda_i}(\infty) = \delta_{\lambda_i, \lambda_0^i},$$

we solve equation (20). Then, the probability for the $m \rightarrow m'$ transition is given by

$$P(m \rightarrow m') = \frac{1}{n - |m|} \sum_{\lambda_1^0 + \lambda_2^0 = m, \lambda_1 + \lambda_2 = m'} \left| C^{(\lambda_1^0)}_{\lambda_1}(\infty) C^{(\lambda_2^0)}_{\lambda_2}(\infty) \right|^2. \quad (22)$$

The factor $n - |m|$ comes from the degeneracy of the initial state. The cross section is obtained by integrating over all the possible values of $b$, i.e.

$$\sigma(m \rightarrow m') = \int_0^{2\pi} \int_0^\infty P(m \rightarrow m') b \, d\phi \, db \quad (23)$$

where $\phi$ is the azimuthal angle of $b$ around the incident direction.

We do not need to calculate the probability amplitudes for all the transitions. By taking the complex conjugate of equation (20), we can easily verify that

$$\left| C^{(-\lambda_0^i)}_{-\lambda_i} \right| = \left| C^{(\lambda_0^i)}_{\lambda_i} \right|$$

and

$$\sigma(-m \rightarrow -m') = \sigma(m \rightarrow m'). \quad (24)$$

The actual cross section $Q$ (in au) is obtained from the reduced cross section $\sigma$ by

$$Q = \frac{3nZ}{2B} \sigma. \quad (25)$$

Once we obtain the function form $\sigma(\beta)$, the dependence of the cross section on the physical quantities $B$ and $Z$ is easily found from the scaling relation of (11) and (25).

3. Calculation and results

In the present paper, we will consider only the special case in which the incident direction of the ion is parallel to the magnetic field. In this case, we can use equations (18) and (21), and the transition probability depends only on $b$ but not on $\phi$. The integration over the angle $\phi$ in (23) provides the factor of $2\pi$. The general case for an arbitrary incident direction will be discussed in a future paper (Sakimoto 1992).
Since the incident direction is parallel to the magnetic field and a linear trajectory is adopted, the quantities characterizing the adiabatic state are expressed as a function of time $\tau$ by

$$X^2 = b^2 + \beta^2 \tau^2$$

$$w_1 = \sqrt{\frac{1}{X_4} + \frac{2\beta \tau}{X^3} + 1} \quad w_2 = \sqrt{\frac{1}{X_4} - \frac{2\beta \tau}{X^3} + 1}$$

$$\alpha_1 = \cos^{-1} \frac{X^3 + \beta \tau}{X^3 w_1} \quad \alpha_2 = -\cos^{-1} \frac{X^3 - \beta \tau}{X^3 w_2}.$$ 

The adiabatic basis (18) depends on both $X$ and $b$. Therefore, we must prepare the adiabatic potential curve and the non-adiabatic coupling for each value of $b$. However, they can be calculated easily in a closed form.

Since the potential curves are given by $w_i \lambda_i$, the zero point of $w_i$ is a crossing point. Setting $w_i = 0$, we have

$$(X^4 - 1)^2 = -4b^2 X^2.$$ 

This says that the potential curves cross with each other only when $b = 0$ and that we have an avoided crossing point at $X \approx 1$ for small impact parameters ($b \ll 1$). Figure 1 shows the adiabatic potential curves $w_1 \lambda_1$ for $n = 3$. There is no noticeable structure when $b \gtrsim 1$. However, we can see the clear avoided crossing at $X \approx 1$ when $b = 0.1$. The potential curves for all $\lambda_i$'s have a common avoided crossing point.

In figure 2, we show the time derivative of the polar angle $\alpha_1$. From (21), this determines the strength of the non-adiabatic coupling. Since the polar angle is a function of $\beta \tau$, the coupling strength is proportional to $\beta$ for given values of $b$ and $X$. Hence, the coupling becomes strong with increasing $\beta$. As the impact parameter increases, the derivative becomes smaller but takes a finite value in a wider range of $\beta \tau$. When $b = 0.1$, it has a localized peak at $\beta \tau \sim -1$ and $0$. The localization at $\beta \tau \sim -1$ comes from the avoided crossing. For the peak at $\beta \tau \sim 0$, the effect of the coupling is not so significant as long as the velocity is low. This is because the difference between the potential curves is also very large at $\beta \tau \sim 0$ (see figure 1).

Taking into account these pictures, we can distinguish the transition mechanism into two types each of which is seen in the collisions at $b > 1$ or $b < 1$:

I) $b > 1$: the transition occurs gradually in a wide range of distances by weak long-range coupling; and accordingly the net transition is mainly the one with $\Delta \lambda_i = 1$.

II) $b < 1$: the avoided crossing determines the transition; and owing to the multi-curve avoided crossing, the transition with $\Delta \lambda_i \geq 1$ is possible.

In figure 3, we show the opacities $b \times P(m - m')$ for $n = 3$ and $\beta = 1$ as a function of the impact parameter. Integrating the opacities gives the cross section. At large impact parameters, the most possible inelastic transition is either of $\Delta \lambda_1 = 1$ or $\Delta \lambda_2 = 1$, which leads to the $\Delta m = 1$ transition. For this reason, the main contribution to the $m = 0 \rightarrow 1$ cross section comes from the type I transition. The avoided crossing is less important in this case. When $\Delta m = 2$, both the $\Delta \lambda_1 = 1$ and $\Delta \lambda_2 = 1$ transitions must occur simultaneously. Hence, at $b > 1$ the opacities are finite but smaller than those of $\Delta m = 1$. Since the collision at $b < 1$ also
Figure 1. Adiabatic potential curves of the Hamiltonian $h_3$ as a function of $\beta \tau$ for $n = 3$. The adiabatic potential curves of $h_2$ can be seen by changing the sign of the horizontal axis.
becomes important, the effect of the avoided crossing is not negligible for $\Delta m = 2$. When $\Delta m > 2$, the long-range coupling can hardly produce this transition, and thereby the opacities are very small at $b > 1$. In this case, the type II mechanism plays a dominant role. We see the oscillatory structure of the opacities at $b < 1$, which is the effect of the avoided crossing. The structure is most pronounced for the $m = -2 \rightarrow 2$ transition. Interestingly, the local minimum or maximum of the oscillation is located at almost the same position for all the transitions. The reason is evident by noticing that all the potential curves have a common avoided crossing point.

Figure 4 shows the cross sections for $n = 3$ plotted against the velocity $\beta$. For a given initial state, the cross section becomes largest when $\Delta m = 1$ since the transition is mainly caused by the long-range coupling. In the approximation of the first order of the dipole interaction $\mathbf{R} \cdot \mathbf{a}/X^2$ (i.e. in the Born approximation), the $\Delta m > 1$ transitions are forbidden. Nevertheless, the cross sections for these transitions have large values. This means that the dipole interaction causes a strong mixing effect. When the collision velocity becomes very high, the Born approximation is satisfactory; and hence the $\Delta m = 1$ transition becomes clearly dominant in the inelastic collision. In table 1, we present the cross sections $\sigma(0 \rightarrow m)$ for $n = 10$. Because of the separability of the Schrödinger equation discussed in section 2.3, the calculation for high $n$ states is not formidable. Transitions of large $\Delta m$ can occur actually through
the multi-curve avoided crossing point. However, since the coupling matrix element (21) has the selection rule of $\Delta \lambda_i = 1$, very large changes in $m$ cannot readily take place. Therefore, the cross section becomes small for the large $\Delta m$ transitions. When $\beta < 1$, the cross sections for the $\Delta m > 2$ transitions are well proportional to $\beta$. This can be considered as the effect of the avoided crossing. To explain this, we invoke the Landau–Zener model (although this model may not be applicable to the
Table 1. Cross sections $\sigma(m = 0 \rightarrow m')$ for $n = 10$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$m'$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
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<tr>
<td>0.2</td>
<td>0.669</td>
<td>0.286</td>
<td>0.168</td>
<td>0.072</td>
<td>0.031</td>
<td>0.008</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>1.526</td>
<td>0.614</td>
<td>0.342</td>
<td>0.141</td>
<td>0.059</td>
<td>0.015</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>2.408</td>
<td>0.926</td>
<td>0.506</td>
<td>0.206</td>
<td>0.089</td>
<td>0.026</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>4.161</td>
<td>1.393</td>
<td>0.711</td>
<td>0.263</td>
<td>0.101</td>
<td>0.023</td>
<td></td>
</tr>
</tbody>
</table>

The present case of the multi-curve avoided crossing. When $b$ is small, the separation between the two adjacent potential curves at $X = 1$ is equal to $b$. Therefore, the Landau–Zener model provides the transition probability by $\exp(-Cb^2/\beta)$ where $C$ is some constant. Integrating the probability times $b$ over the whole range of $b$, we can find that the resulting cross section is proportional to $\beta$. At very low velocities, even the $\Delta m = 1$ and $2$ transition can occur only through the avoided crossing point. In the low-velocity region, thereby, the cross sections for $\Delta m = 1$ and $2$ also become proportional to $\beta$ (see figure 4). When $\beta$ becomes large, the transition is not localized at the avoided crossing point so that the proportionality relation breaks down for all the transitions.

![Figure 5](image-url)

Figure 5. Scaled cross sections $\sigma(m = 0 \rightarrow 1) \times n^{-1}$ and $\sigma(m = 0 \rightarrow 3) \times n^{-1.125}$ plotted against $\beta \times n^{-1/2}$ for $n = 2 \sim 10$.

Figure 5 shows the scaled cross sections for the transitions of $m = 0 \rightarrow 1$ and $3$. The cross sections times $n^{-q}$ are plotted against $\beta/\sqrt{n}$. The value of $q$ is taken as $q = 1$ for $m = 0 \rightarrow 1$ and $q = 1.125$ for $m = 0 \rightarrow 3$. Except for some of the smallest $n$, the scaled curves almost coincide with each other. Hence, we have the scaling law for $n$ which is valid at least for large $n$. For other transitions, we can also obtain the same result by taking an appropriate value of $q$; for example, $q \approx 1$ for $m = 0 \rightarrow 2$ and $q = 1.25$ for $m = 0 \rightarrow 5$, etc. From this result, the actual cross section $Q$ is almost proportional to $n^{q+1}$ for a given value of the collision energy in units of $n^2$. For small $\beta$ furthermore, since $\sigma \propto \beta$, we can easily find that $Q \propto n^q \sqrt{Z/B^3v}$. 

The meaning of the scaling parameters, $n^q$ ($q \sim 1$) for $\sigma$ and $n^{1/2}$ for $\beta$, is roughly explained as in the following. We may define the interaction range $X_0$ by setting the magnitude of the dipole interaction to be equal to the energy difference of the potential curves $\Delta \epsilon$. That is $|\alpha|/X_0^2 \sim \Delta \epsilon$. Since $\Delta \epsilon$ is nearly independent of $n$ and $|\alpha| \sim n$, $X_0^2$ (and thus $\sigma$) has nearly a linear dependence on $n$. The cross section has a maximum value when the velocity ($\beta_0$) satisfies the condition $\Delta \epsilon X_0 / \beta_0 \sim 1$. This gives $\beta_0 \sim \sqrt{n \Delta \epsilon}$.

![Figure 6. Cross sections $Q(m = 0 \rightarrow 1)$ in cm$^2$ as a function of $B$ in Tesla for $H^+ + H(n = 3)$ at the relative collision energies of 0.01, 0.1 and 1.0 eV.](image)

In figure 6, as an actual example, we show the magnetic field dependence of the cross section $Q(0 \rightarrow 1)$ for the collisions between a proton and a hydrogen atom in the $n = 3$ state at some relative collision energies $E = 0.01$, 0.1 and 1.0 eV. The results have been obtained by using the scaling law (11) and (25). The cross section $Q$ is monotonic with respect to $B$. It diverges as $B \rightarrow 0$. This is because the $(l, m)$ sublevels are degenerate in the absence of external fields and, furthermore, the transition occurs through the dipole interaction. As shown before, the cross section $Q$ decreases as $B^{-3/2}$ for large $B$ (i.e. for small $\beta$). For a given energy $E$, we can obtain a functional relation of $Q_E(B_{E'})$ as shown in figure 6. When the energy is changed as $E \rightarrow E'$, the scaling law says

$$Q_{E'}(B_{E'}) = \frac{E}{E'} Q_E \left( \frac{E}{E'} B_{E'} \right).$$

(26)

4. Discussion

For application to real plasmas, the incident ion does not always approach the hydrogen atom along a magnetic field. For an arbitrary incident direction, the transition
probability also depends on the orientation of the impact parameter vector. The integration over the angle \( \phi \) must be made numerically in equation (23). Furthermore, the cross section has a dependence on the angle \( \chi \) spanned by the incident direction and the magnetic field. For application to real plasmas, the cross section should be averaged over this angle. These steps require much more computational effort. A preliminary calculation shows that the transition probability strongly depends on the angle \( \phi \) but the cross section weakly depends on the angle \( \chi \). As a result, the real cross section would not be largely different from the one obtained in the present calculation. Details for the general case will be reported in the next paper (Sakimoto 1992).

If the photon emission of the excited atom is much faster than the collision process that produces disalignment, we can neglect the disalignment effect. The collision process becomes important when the number density of the ion exceeds a critical value. The lifetime for the 3p state of hydrogen is \( \sim 5 \times 10^{-9} \) s; and from figure 6 we have \( Q \approx 5 \times 10^{-10} \) cm\(^2\) when \( B = 0.1 \) T and \( E = 1 \) eV. In this case, the critical density is \( \sim 10^{15} \) cm\(^{-3}\). Since the cross section is larger and the lifetime is longer for higher principal quantum numbers \( n \), the critical density becomes much smaller for higher \( n \).

Acknowledgment

The author would like to thank Dr Fumihiro Koike for providing useful comments.

Appendix

In the present calculation, we have assumed that the motional Stark effect is negligible. This assumption is not always satisfied in many plasmas. We have introduced it to make the calculation tractable. Here, we discuss its validity.

We consider that the hydrogen atom moves across a magnetic field with the velocity \( V \) in atomic units (\( V \perp B \)). Then, an electric field is induced as \( F_M = 2V \times B = 2VB\hat{F}_M \). Adding this term to the Hamiltonian \( (10) \) and using the relation \( (2) \), we have in the reduced units

\[
h = w_1 \cdot j_1 + w_2 \cdot j_2 \tag{A.1}
\]

where

\[
w_1 = \hat{B} - 3nV\hat{F}_M + \frac{1}{X^2} \hat{\mathbf{R}} \quad \text{and} \quad w_2 = \hat{B} + 3nV\hat{F}_M - \frac{1}{X^2} \hat{\mathbf{R}}. \tag{A.2}
\]

From this equation, the motional Stark effect is negligible if \( 3nV \ll 1 \). Hence, the present study is applicable for \( 3nV \ll 1 \).

It should be noted that equation \( (A.1) \) is valid even if an arbitrary electric field is added. Therefore, we find that the Hamiltonian is still separable for the collisions in the presence of crossed electric and magnetic fields. However, the axial symmetry around the magnetic field is evidently lost even in the limit of \( X \rightarrow \infty \), and the determination of the quantization axis cannot be made straightforwardly.
References

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